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The Punctured Plane is Isomorphic to the Unit Circle

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This note shows that the multiplicative group of non-zero complex numbers and the multiplicative group of complex numbers of absolute value 1 are isomorphic as groups.

Let K^* denote the multiplicative group of non-zero complex numbers, and let T denote the proper subgroup of K^* of all complex numbers of modulus 1. *We shall see that K^* and T are isomorphic as groups.* First note that K^* and T are divisible abelian groups, i.e. each equation $x^n = b$ has a solution for every positive integer n and for every element b of the group. Theorem 19.1 of [3] or Theorem 9.13 of [4] shows that every divisible group is isomorphic to a direct sum of copies of Q, the rationals, and copies of quasicyclic groups $\sigma(p^\infty)$ for various primes p. Let P denote the set of all prime natural numbers; thus we have

$$K^* \cong \sum_a Q \oplus \sum_{p \in P} \sum_{m_p} \sigma(p^\infty),$$

and

$$T \cong \sum_b Q \oplus \sum_{p \in P} \sum_{n_p} \sigma(p^\infty).$$

We shall show that the cardinal numbers $a = b$ and each $m_p = n_p$. Certainly each $m_p \geq n_p \geq 1$. If some $m_p > 1$, then $\sigma(p^\infty) \oplus \sigma(p^\infty)$ is isomorphic to a subgroup of K^*. Hence, if $\sigma(p^k)$ is the cyclic group of order p^k, for some integer $k > 0$, then $\sigma(p^k) \oplus \sigma(p^k)$ is isomorphic to a subgroup of K^*. So K^* has p^{2k} elements satisfying $x^{p^k} = 1$, contrary to the fact that there are exactly p^k elements in K^* satisfying this equation. Hence $m_p = n_p = 1$.

Now $a = b = 2^{\aleph_0}$, the first uncountable cardinal, since otherwise K^* and T would not have the correct number of elements. So K^* and T are each isomorphic to

$$\sum_{2^{\aleph_0}} Q \oplus \sum_{p \in P} \sigma(p^\infty).$$

It is also of interest to note that K^* and T are topological groups with the usual topologies. But T is compact and K^* is not.

For more information on groups isomorphic to proper subgroups see [*1*] and [*2*].

References

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